Mathematics Triple Record

Adam, Cecilia, Michael, Sam

August 3, 2012

Sunday, July 29

We began class by going over problems from previous problem sessions. Bradley began by presenting his proof of the statement: if S is linearly independent, then $A \subset S$ is also linearly independent. To do this, he did a simple proof by contradiction showing that there cannot be non-zero solutions to $\sum_{i=1}^{k} \lambda_i x_i = 1 \ \forall x_i \in A$ without there being a contradiction. This proof showed that linear dependency does not transfer to subsets while linear independency does. Eric then presented his proof: $\exists x_i \in S, S = \{x_1, \ldots, x_i\} |\ lin(S) = lin(\{x_1, x_2, \ldots, \hat{x_i}, \ldots, x_k, y\}), y \in lin(S)$. We briefly discussed the concept of regularity. Bradley conjectured that regular meant that all ℓ-faces could be isometrically mapped to all other ℓ-faces for all ℓ at the same time, with $\ell \leq d$. We then showed that an affine hull of a set S is is the intersection of all affine subspaces containing S. We then continued to prove this for linear and convex hulls as well. Bradley then conjectured that $lin(S) = aff(S \cup \{\mathbf{O}\})$, which was proved on Tuesday. Michael then presented his proof that if $z \in conv(\{x, y\})$, then $\mathrm{K}(x, 1) \cap \mathrm{K}(y, 1)$ was a subset of $\mathrm{K}(z, 1)$. After fiddling with lots of notation, we established that dot product notation reduces the need for large numbers of sigmas.

Monday, July 30

We began the week as we usually do, using a diagram of skybridges to determine who was to take notes on which day. After we all tried to determine to which day our names corresponded, we launched into a study of convex hulls and polytopes. sarah-marie allowed the class to split up into 4 groups, and each group was given some number of coordinates. Each group was directed to compute the convex hull of its points, determine the f-vector, shapes of faces, etc. Here, we were introduced to the B function, which, for some polytope P, is $\{x \mid x \in \mathbb{R}^d, \langle x, y \rangle \leq 1 \ \forall y \in P\}$. Immediately, several conjectures were made, and the one with the most support was that $B(chicken) = squid$ (up to some potential modifications), which was conjectured by Barry. Later, more conjectures followed: An algorithm by Bradley, and two conjectures by Ethan and by Michael/Sam:

- Bradley's Algorithm for constructing B: $\forall \varphi \in \partial(P)$ with $\varphi = (x_1, x_2, \ldots, x_d)$, construct a halfspace A_α through $\{(1/x_1, 0, \ldots, 0), \ldots (0, 0, \ldots, 1/x_d)\}$. B will be equal to the intersection of all A_α.
- Ethan's Conjecture (The Turq. Thm, proved on Tuesday): $B(T_1 \cup T_2 \cup \cdots \cup T_k) = \bigcap_{i=1}^{k} B(T_i)$.
- Michael/Sam's Assumption: If $P \subset Q, B(Q) \subset B(P)$ (proved later).

In the afternoon, we explored some linear programming, as we had seen with the horse video from the week before. We discussed maximization and minimization of functions based on certain parameters or restrictions. We also looked at the dual of a linear program, which is different from the dual of a polytope.

Tuesday, July 31

Conjectures:

- The intersection of two ℓ-substructures is an ℓ-substructure.
- $B(Chicken) = Squid$ iff $\mathbf{O} \in \text{int}(Chicken)$.
- $B(P) = B(\partial(P))$.

Proved:

- $V =$ set of 0-faces of $conv(S)$, $V \subset S$.
- If $\langle x, y \rangle = 0$, then x, y are linearly independent.
- $B(P)$ is always convex.

Wednesday, August 1

We started off the day with Bradley presenting how a polytope is a subset of the intersection of half-spaces. Then Adam showed that $P \ni \mathbf{O} \Rightarrow B(B(P)) \supseteq S$. He also came up with an algorithm to find a half-space that does not contain any point p outside of a convex set, thereby showing that the intersection of all half-spaces containing a polytope $P = conv(S)$ is contained in C. That is, we showed that: $conv(S) = \bigcap\limits_{\substack{i \in I \\ K \supset S}} K(a_i, c_i)$. But also something stronger: $conv(S) = \bigcap\limits_{\substack{i \in I \\ K \supset S \\ K \text{ supp } conv(s)}} K(a_i, c_i)$.

Nate had some **BIG QUESTIONS**:

- Is a polytope equal to the convex hull of its vertices?
- Can a polytope always be defined by a finite number of inequalities?
- What is going on with $B(P)$?

Questions about Faces:

- Are they polytopes?
- How do they intersect?
- What are possible f-vectors of polytopes?

Conjecture: $conv(S) = \bigcap_{i \in I} K(a_i, c_i)$, with $\partial(K) \supset F=$ facet of $P = conv(S)$, a polytope.

Wednesday Night and Thursday, August 2

First, we had Michael explain what happened when a vertex of a cube was moved slightly. Through this he showed us that the f-vector cannot determine the polytope, and that multiple polytopes can have the same f-vector. Sam then proved that any set of points that were pairwise orthogonal would be linearly independent. New conjectures were made:

- elface of a d-polytope is an ℓ-polytope (now proven!)
- elface L of P can be expressed as $conv(S), S \subset V$ where V is the set of vertices of P (now proven!).
- if L_1, L_2 are substructures, $L_1 \cap L_2$ is a substructure.

Vlad shared what he noticed. He said that the dual of a d-pyramid is the pyramid of the dual of P_{d-1} (the base). Michael and Ethan collectively showed that $B(S) = B(conv(S))$. We then worked with face lattices, a new combinatorial way to represent polytopes. Simple polytopes have every vertex in exactly d facets. Simplicial polytopes have $(d-1)$-simplices as all their facets.

Friday, August 3

1. Bradley showed that an ℓ-face $L = conv(A)$ of a polytope is also a polytope, using double inclusion to show that $P = conv(S) \Rightarrow \exists A \subset S \mid conv(A) \in L$ by double inclusion, that $conv(A) \in \mathrm{H}(a_i, c_i)$ supp P and $conv(A) \subset conv(S)$. He then used contradiction to show that $\nexists x \in H \cap P, x \notin conv(A)$.

2. Then Sam proved that a polytope $P = conv(V)$, V is the set of vertices of P, by showing that any other point in the polytope can be written as a convex combination of the vertices, but each vertex must be written as the convex combination of just itself.

3. Then Cecilia showed that an ℓ-face L can be written as $conv(B)$, B is a subset of V. She combined (1) and (2) to show that if a polytope $P = conv(V)$ such that V is the set of vertices, and $L = conv(Q)$ such that Q is the set of Ls vertices, then $Q \subset V$ (by (2)).

4. Finally, Barry and Adam showed that $B(P) = B(\partial(P))$ because for any point $a_i \in P, a_i = \lambda_i x_i$.