Mathematics Triple Record

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Isometries

A very nice film-of-a-film introduced isometries. It was explained in various ways how to move points, lines, and polygons around a plane while preserving distance. Rotations, reflections, and translations were all types of isometries although it was proved various times that all isometries can be composed of reflections. Further exploration of isometries followed the day after; particularly their clump-ness. Using the various components from the definition of a clump, it was proved that the set of isometries is indeed a clump. The fact that isometries can all be composed of reflections was shown to be helpful in proving that isometries have an inverse. This led to further work on generating clumps. The question was brought up about isometries in the third dimension, but we have little reported progress on this matter at the present.

Clump Actions

A lot of the work that has been seen with clumps fits into a nice little category called clump actions. Clump actions happen when a clump acts on an underlying set. Examples of this that have come up are rotations and flips of polygons, with the rotations/flips being the clump and the points of a polygon being the set. Also brought up were rotations of a cube.

In an entertaining Number Devil story accompanied by a dramatic interpretation by A., B., C., D., and E., it was discovered that, besides the fact that when Bradley gets angry she has the tendency to bite and scratch, the actions of changing permutations of n people comprise a clump action. Possibly even more amazing was the finding that this whole clump action could be described just by switching one pair of letters at a time. This leads into a whole new field of generating clumps. Meanwhile, the students went about trying to define functions and notate clump actions. Some very suitable notations arose, leading to discussion of combining clump actions. In conclusion, we discovered that clump actions can be reduced to compositions of fewer, simpler elements.

Infinity

Through the use of the Hilbert Hotel story, a thrilling tale detailing the exploits of Mr. and Mrs. Smith, we learned about the concept of infinity. In an infinite set, every number can be moved up by one to create an additional spot. To create an infinite amount of spaces, every number can be doubled (or the equivalent of doubling in the set), freeing up the first spot, third spot, and so on, ad infinitum. This property of infinity can be used to create an infinite amount of infinite openings as well as to give an infinite amount of people two sausages each when each person has been provided with only one sausage. We were also able to prove that the set of the sets in the set of natural numbers is larger than the set of natural numbers.

The students learned that the requirements for a function were well-defined, one-to-one, and onto. It was noted that two sets are the same size if there is a one-to-one correspondence between the sets. After further thought, it was discovered that a function needs to be one-to-one and onto in order to be considered in one-to-one correspondence.

Surfaces

A surface is a compact (closed and bounded), smooth, connected, 2-manifold without boundary. One example of a surface is a sphere (denoted here as S). Another is a torus. Some surfaces can be represented with a 2n-gon with edges that have been labeled with orientation. A polygonal representation can be ‘desnouted’ to split it into 2 separate representations. When desnouted, the line of desnouting becomes an oriented edge for both polygons. These 2 polygons can be resnouted at any matching edge. The two 2-gons are P (when the lines are oriented towards different vertices) and S (when the lines are oriented towards the same vertex). Snouting S with another polygon has no effect on the surface the polygon represents. All 4-gons can be either reduced to P, S, the snout sum of 2 Ps, or a torus. When two linked edges are oriented at the same vertex, they can be reduced to a single vertex. Because of this and the aforementioned snout rules, any polygonal representation of a surface can be reduced to either, P, S, a torus, or the snout sum of the aforementioned surfaces.

Knots

A knot is a closed polygonal ffrag in R^3 where every vertex has degree 2. A link is a finite union of disjoint knots. Knot diagrams are 2 dimensional representations of a knot. A single knot can have infinitely many diagrams because it exists in 3-D space. 2 knots are equivalent if they have the same diagram. Reidemeister moves are a set of three different moves that allow the transformation of one knot diagram into an equivalent diagram. The three Reidemeister moves are unlooping, shifting a length off of another length, and shifting a length to the opposing side of a crossing. An invariant is something a knot has that never changes, regardless of how it is represented. One such invariant is the minimum number of crossings. While 2 different invariants are indicative of 2 differing knots, 2 identical invariants do not necessarily imply 2 identical knots. Knots can be given orientations. This helps when determining a knot’s polynomial invariant. A crossing can be modified so that the strand going to either the right or left is on top, or to even eliminate the crossing entirely; only one crossing can be changed this way at a time. The knot’s polynomial invariant is determined by deriving the invariant for the original position from the two other options from modifying the selected knot crossing. All knots can eventually be broken down into the unknot (a circle), which has a value of 1. It is from the unknot that all knots derive their polynomial invariants.

Cayley ffrags

Cayley ffrags are used to represent clumps. An example of a cyclic clump is the set of square rotations. It can be rotated by 0/360°, 90°, 180°, or 270°. Each of these elements can be reached by rotations of 90°. A Cayley ffrag of this clump would be a square where the vertex 0/360° connects to 90° which connects to 180° which in turn connects to 270° which finally connects to 0/360°.

nth roots of unity:

The nth roots of unity are an example of a type of Cayley ffrag. When $z^n = 1$, the possibilities of what z can be determines and generates the set of the nth roots of unity.

Clump Generators

We can notate any clump by showing what generates the clump elements and how the generators interact with each other as a presentation: <(generators) | (relations)>. For example, the nth roots of unity can be represented as $\langle w \mid w^n = e \rangle$, and the clump action comprised of rotations and reflections of a triangle with labeled edges can be represented as $\langle a,b \mid a^3 = e, b^2 = e, ab = ba^{-1} \rangle$.

Cutting a Watermelon

Let C be a 2-dimensional cut. Each intersection between C and another 2-D cut not parallel to C is a line. Then, each 2-D piece in C bounded by intersection lines corresponds to cutting the adjacent 3-D piece into two, i.e.: each 2-D piece in C bounded by intersection lines adds 1 to the total number of 3-D pieces. The maximum number of bounded 2-D pieces formed by m cuts is $h_2(m)$. So, adding an nth 2-D cut to a set of (n-1) 2-D cuts can create at most $h_2(n-1)$ new pieces. Thus, $\mathbf{h_3(n)=h_3(n-1)+h_2(n-1)}$.

The conditions for generating the maximum number of cuts are:
- no parallel cuts
- cuts cannot contain already existing intersection line
- three cuts cannot intersect at one point

Cutting a general k-dimensional fruit (hyperhyperhyper......hyperfruit?)

Let C be a (k-1)-dimensional cut. Each intersection between C and another (k-1)-D cut not parallel to C is a (k-2) plane. Then, each (k-1)-D piece in C bounded by intersection (k-2)-planes corresponds to cutting the adjacent k-D piece into two, i.e.: each (k-1)-D piece in C bounded by intersection (k-2)-planes adds 1 to the total number of k-D pieces. The maximum number of bounded (k-1)-D pieces formed by m cuts is $h_k\text{-}1(m)$. So, adding an nth (k-1)-D cut to a set of (n-1) (k-1)-D cuts can create at most $h_k(n-1)$ new pieces. Thus, $\mathbf{h_k(n)=h_k(n-1)+h_{k-1}(n-1)}$.

The conditions for generating the maximum number of cuts are:
- no parallel cuts
- no (j+1)-cuts intersect at a (k-j)-dimension unit

A Bent Math: The Batman Counting Principle

Batman: "The documentary about me isn't just about me. It's also about the hardworking men and women of the Gotham police department. So, I was planning to buy them donuts. I needed to buy 24 donuts, so I went to Dunkin Donuts, and bought 12 chocolate donuts, and 12 donuts with sprinkles. But after counting them, Alfred told me that I only had 20 donuts! Impossible! Then, I realized that some donuts were chocolate and sprinkled donuts. So, how many donuts are chocolate and sprinkled?"

<u>Donut count:</u>
12 chocolate donuts
12 sprinkled donuts
20 donuts total

|chocolate| + |sprinkled| - |chocolate sprinkled| = |total|
12 + 12 - |chocolate sprinkled| = 20
|chocolate sprinkled| = 4
Thus, four donuts are chocolate and sprinkled.

<u>We now introduce the Batman Counting Principle for two sets:</u>
$|A| + |B| - |A \cap B| = |A \cup B|$

Batman: "Gotham has many supervillains. And to keep Gotham safe, I decided to strike them at their henchmen. The Joker employs 21 henchmen. The Riddler employs 22 henchmen. The Penguin employs 24 henchmen. But some henchmen can work for multiple supervillains because henchmen, you know, have to make ends meet. 7 henchmen work for both Joker and Riddler, 9 henchmen for Joker and the Penguin, and 11 henchmen for both Riddler and the Penguin. There are 43 total henchmen. So, how many henchmen work for all three supervillains?"

Let A = {set of henchmen working for Joker}, B = {henchmen working for the Riddler} and C = {henchmen working for the Penguin}.

<u>The Batman Counting Principle for three sets is as follows:</u>
$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| = |A \cup B \cup C|$

Now we solve:
$|A \cap B \cap C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |C \cap A| - |A| - |B| - |C|$
$= 43 + 7 + 11 + 9 - 21 - 24 - 22 = 3.$
Thus, three henchmen work for all three supervillains.

<u>General form of Batman Counting Principle:</u>
Given sets $S_1, S_2 \ldots, S_n$,

$$|S_1 \cup S_2 \cup \ldots \cup S_n| = \sum_{a=1}^{n} |S_a| - \sum_{1 \le a < b \le n} |S_a \cap S_b| + \sum_{1 \le a < b < c \le n} |S_a \cap S_b \cap S_c| - \ldots + (-1)^{n-1} |S_1 \cap S_2 \cap \ldots \cap S_n|$$

Coloring!

We must create an arbitrary pretty picture by connecting a set amount of balloons to each other with strings. We must color in the balloons and strings according to the following rules:
1. Balloons connected by strings cannot be the same color
2. Strings that share a balloon cannot be the same.

We now investigating the lower and upper bounds for the number of colors needed to color in
a) balloons and b) strings:

1. Lower bound for number of colors needed for…
…Balloons: A lower bound for the highest n such that K_n is included in the ffrag of balloons/strings.
…Strings: Highest vertex degree

2. Upper bound for number of colors needed for…
…Balloons: number of balloons
…Strings: number of strings.

Problems Recently Posed

In every issue of each of the Mathematical Association for America (MAA) journals, there is a section of problems and solutions. The problems are accompanied by a deadline by which proposed solutions must be submitted; the solutions are to problems from prior issues of the journal.

The problems below were selected because they have few (if any) prerequisites, and they are not super-boring. In fact, they might be considered interesting.

If you think you have a solution, please ask one of the instructors to review it for correctness; and, if your solution seems to be correct, ask for assistance in submitting the solution.

(1) This problem appeared in *the American Mathematical Monthly*. The solution deadline is July 31st, 2012. (**Hey! That's in only a week!**)

11632. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Dan Schwarz, Bucharest, Romania.* Let n be a positive integer, and write a vector $\mathbf{x} \in \mathbb{R}^n$ as $(x_1, \ldots, x_n)$. For $\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ let

$$[\mathbf{x}, \mathbf{y}]_{\mathbf{a},\mathbf{b}} = \sum_{1 \le i, j \le n} x_i y_j \min(a_i, b_j).$$

Show that for $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^n$ with nonnegative entries,

$$[\mathbf{x}, \mathbf{x}]_{\mathbf{a},\mathbf{a}} \cdot [\mathbf{y}, \mathbf{z}]^2_{\mathbf{b},\mathbf{c}} + [\mathbf{y}, \mathbf{y}]_{\mathbf{b},\mathbf{b}} \cdot [\mathbf{z}, \mathbf{x}]^2_{\mathbf{c},\mathbf{a}} \le [\mathbf{x}, \mathbf{x}]^{1/2}_{\mathbf{a},\mathbf{a}} \cdot [\mathbf{y}, \mathbf{y}]^{1/2}_{\mathbf{b},\mathbf{b}} \cdot [\mathbf{z}, \mathbf{z}]_{\mathbf{c},\mathbf{c}} \cdot \left([\mathbf{x}, \mathbf{x}]^{1/2}_{\mathbf{a},\mathbf{a}} \cdot [\mathbf{y}, \mathbf{y}]^{1/2}_{\mathbf{b},\mathbf{b}} + [\mathbf{x}, \mathbf{y}]_{\mathbf{a},\mathbf{b}} \right).$$

(2) This problem appeared in *the American Mathematical Monthly*. The solution deadline is September 30th, 2012.

11646. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let ABC be an acute triangle, and let A_1, B_1, C_1 be the intersection points of the angle bisectors from A, B, C to the respective opposite sides. Let R and r be the circumradius and the inradius of ABC, and let R_A, R_B, R_C be the circumradii of the triangles AC_1B_1, BA_1C_1, and CA_1B_1, respectively. Let H be the orthocenter of ABC, and let d_a, d_b, d_c be the distances from H to sides BC, CA, and AB, respectively. Show that

$$2r(R_A + R_B + R_C) \ge R(d_a + d_b + d_c).$$