Grünbaum Colorings of Toroidal Triangulations

Michael O. Albertson[*], Hannah Alpert[†], sarah-marie belcastro[*‡], and Ruth Haas[*]

April 27, 2008

Abstract

We prove that if G is a triangulation of the torus and $\chi(G) \neq 5$, then there is a 3-coloring of the edges of G so that the edges bounding every face are assigned three different colors.

Keywords: embedding, edge coloring, Grünbaum coloring, Grünbaum conjecture, triangulation

1 Introduction

Our story begins with Kempe's famous (but false) proof of the Four Color Conjecture (4CC). Subsequently Tait claimed, but did not publish, another proof based on the (false) belief that every cubic 3-connected planar graph contains a Hamilton cycle. Tait did understand that a 4-coloring of the faces of such a graph is equivalent to a 3-coloring of its edges. See [8] for details of this fascinating story. If one dualizes Tait's observation, one obtains an equivalent version of the 4CC for planar graphs, one with a different notion of edge coloring. We say that a 3-coloring of the edges of an embedded

[*]Department of Mathematics and Statistics, Smith College Northampton, MA 01063 USA

[†]340 Fox Drive, Boulder, CO 80303 USA

[‡]corresponding author: email smbelcas@toroidalsnark.net, phone 413-341-1373, fax 413-585-3786

triangulation is a *Grünbaum coloring* if every facial triangle is incident with three different colors. Note this places no constraint on the colors assigned to the edges of either a separating or a non-contractible 3-cycle, nor does it require incident edges to receive different colors. In 1968 Grünbaum offered a far-reaching generalization of the 4CC: every triangulation of every orientable surface has a Grünbaum coloring [12]. Since the first proof of the Four Color Theorem (4CT) [5, 6, 16], Grünbaum's conjecture has gained increasing notoriety [7].

The purpose of this paper is to prove the following.

Theorem 1.1. *If G is a triangulation of the torus with $\chi(G) \neq 5$, then G has a Grünbaum coloring.*

Recently Kochol discovered counterexamples to Grünbaum's conjecture on every orientable surface S_g with $g \geq 5$ [14]. Our result contrasts with Kochol's. The following problems remain open: Does every 5-chromatic toroidal triangulation have a Grünbaum coloring? For $2 \leq g \leq 4$, does every triangulation of S_g have a Grünbaum coloring? Does every *locally planar* triangulation of S_g (those embedded with large edge width) have a Grünbaum coloring?

Our proof is a divide-and-conquer argument and consists of a sequence of lemmas. In the second section we show that if $\chi(G) \leq 4, \chi(G) = 7$, or G is 6-regular, then G has a Grünbaum coloring. In the third section we treat 6-chromatic graphs whose critical 6-chromatic subgraphs are not K_6. Finally, in the fourth section we complete the proof by examining those 6-chromatic toroidal triangulations that contain K_6.

2 First Results

Our first result is folklore. Many who think about Grünbaum's conjecture would dualize the statement and reproduce Tait's proof. We prefer a direct argument inspired by graph homomorphisms.

Lemma 2.1. *If G is a triangulation of any surface and $\chi(G) \leq 4$, then G has a Grünbaum coloring.*

Proof. Fix a vertex 4-coloring of G and a standard edge 3-coloring of K_4. Fix a vertex 4-coloring of K_4 by using the labels of the vertices as colors. Suppose $e = uv \in E(G), \mathrm{col}(u) = i$, and $\mathrm{col}(v) = j$; color e with $\mathrm{col}(ij)$. Any

3-cycle in G corresponds to a triangle in K_4. Since the edge coloring of K_4 is Grünbaum, so is the edge coloring of G. □

Lemma 2.2. *Every* 6*-regular toroidal graph has a Grünbaum coloring.*

Proof. It is an immediate consequence of Euler's formula that a 6-regular toroidal graph is a triangulation. Altshuler has shown that every 6-regular triangulation of the torus can be realized as a rectangular grid with diagonals in which the left and right sides are identified and the top and bottom sides are identified with a twist [4]. Every face in such a graph has one vertical edge, one horizontal edge, and one diagonal edge. The result follows immediately upon realizing that "vertical", "horizontal", and "diagonal" are colors. □

Lemma 2.3. *Suppose H is a triangulation of a surface S that has a Grünbaum coloring. If G is a triangulation of S that contains H as a subgraph, i.e. G is a refinement of H, then G has a Grünbaum coloring.*

Proof. Pick a Grünbaum coloring of H. Transfer this edge coloring to G. The edges of G that are not yet colored live inside triangular faces of H. For each such triangular face of H, consider the triangulation of the plane generated by the vertices on and inside this face. Since this is a planar triangulation, it has a Grünbaum coloring by the 4CT. A permutation of the colors in this Grünbaum coloring will make it agree with the Grünbaum coloring of H. Since this can occur independently for every face in H, G will have a Grünbaum coloring. □

Lemma 2.4. *Every 7-chromatic toroidal triangulation has a Grünbaum coloring.*

Proof. Any toroidal graph with chromatic number 7 must contain K_7 as a subgraph [9]. Since K_7 is 6-regular, Lemmas 2.2 and 2.3 finish the proof. □

3 Six-Chromatic Toroidal Triangulations, I

The proof that 6-chromatic triangulations of the torus have Grünbaum colorings is, in outline, much like the proof of Lemma 2.4. However, the details are tricky and will occupy the remainder of this paper. Our proof relies on the fact that critical 6-chromatic toroidal graphs are classified. Clearly K_6 is one such graph. The others are described below.

Given graphs G_1 and G_2, the *join* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph obtained by taking vertex-disjoint copies of G_1 and G_2 and adding every edge uv where $u \in G_1$ and $v \in G_2$. It is straightforward to see that if G_1 is a critical r-chromatic graph and G_2 is a critical s-chromatic graph, then $G_1 + G_2$ is a critical $(r+s)$-chromatic graph. Thus $C_3 + C_5$ is a critical 6-chromatic graph.

Let H_7 denote the graph obtained by applying the first step of the Hajós construction [13] to two vertex-disjoint copies of K_4. Since the Hajós construction preserves criticality, H_7 is a critical 4-chromatic graph. Consequently, $H_7 + K_2$ is a critical 6-chromatic graph. H_7 is shown in Figure 1.

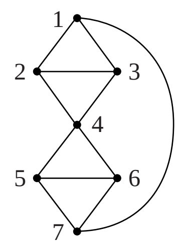

Figure 1: H_7

More than thirty years ago, Albertson and Hutchinson [1, 2] discovered a 6-regular, critical 6-chromatic toroidal graph and named it J. It was later realized that J is isomorphic to C_{11}^3, the cube of the 11-cycle. In [3] Albertson and Hutchinson gave a list of critical 6-chromatic graphs including $K_6, C_3 + C_5$, and $H_7 + K_2$, but mistakenly believed the list was missing an additional critical graph on nine vertices. Fourteen years later Thomassen proved the following.

Theorem 3.1 ([17, 15])**.** *If G is a 6-chromatic toroidal graph, then G contains exactly one of $K_6, C_3 + C_5, H_7 + K_2$, or C_{11}^3.*

Given G, a 6-chromatic triangulation of the torus, we will produce an edge coloring of its critical 6-chromatic subgraph that extends to a Grünbaum coloring of all of G. If a face of a particular embedding of the critical 6-chromatic subgraph is 3-sided, and the three boundary edges of this face are assigned different colors, then the interior of this triangular face has a

Grünbaum coloring of its interior. The argument is identical to the proof of Lemma 2.3.

Lemma 3.2. *If G is a 6-chromatic toroidal graph that contains C_{11}^3, then G has a Grünbaum coloring.*

Proof. This follows immediately from Lemmas 2.2 and 2.3. □

If G is a 6-chromatic toroidal graph that does not contain C_{11}^3, the proof that G has a Grünbaum coloring is complicated by the fact that the critical 6-chromatic subgraph does not embed as a triangulation of the torus.

Lemma 3.3. *$H_7 + K_2$ is uniquely embeddable in the torus.*

Proof. Gagarin et. al. have shown that all embeddings of a graph G on the torus can be found by considering a spanning θ-subgraph of G [11]. (A θ-graph is composed of two vertices of degree 3 joined by three internally disjoint paths.) There are three labeled embeddings of a θ-graph on the plane, each with a different path in the middle. Similarly, there are three labeled embeddings of a θ-graph on a cylinder. There is only one labeled 2-cell embedding of a θ-graph on the torus. These seven cases correspond to possible toroidal embeddings of a spanning θ-subgraph of G. In each case, the remaining edges of G can be added one by one as forced by the embedding or to make appropriately sized faces. In all but one case, this process leads to a contradiction.

Denote the vertices of H_7 by $\{1, 2, \ldots, 7\}$ as in Figure 1 and the vertices of K_2 by $\{a, b\}$. Choose the spanning θ-subgraph G_θ to have $\{a, b\}$ as its vertices of degree 3. Let the three paths be $a4b$, $a213b$, and $a576b$. Euler's formula implies that one face will be a quadrilateral and the rest triangles. We provide a few examples of the process of ruling out potential embeddings.

Case I. Embed G_θ in the plane with $a576b$ as the middle path. The placement of the edges 45, 46 and 17 is now forced. The edge $5b$ cannot be placed without crossings.

Case II. Embed G_θ on the cylinder with paths $a213b$ and $a4b$ homotopic. Consider the possible placements of the edge ab. If it is also placed homotopic to the paths $a213b$ and $a4b$ then either one of the edges 17 or 46 is immediately blocked or the edges 24 and 34 are forced and the edge 17 cannot be placed. If ab is not homotopic, then there are 2 other cases and several subcases, all of which lead to contradictions.

Case III. Embed G_θ in the plane with $a4b$ as the middle path. The placement of the edges incident to 4 is forced. This produces two quadrilateral faces, $(1,3,4,2)$ and $(4,5,7,6)$. There is only one quadrilateral face in the embedding of G, so at least one of the edges 23 or 56 must be placed in a quadrilateral. Without loss of generality, assume 23 divides the quadrilateral $(1,3,4,2)$. The remaining edges are now added to create triangular faces. This produces the unique embedding. The other cases are more involved but follow similar strategy. □

Lemma 3.4. *If G is a triangulation of the torus that contains H_7+K_2, then G has a Grünbaum coloring.*

Proof. The unique embedding of $H_7 + K_2$ contains one quadrilateral face, say Γ. We construct a triangulation of the plane by cutting Γ and its interior out of the torus and placing it in the plane. We then create a new vertex, say u, in the unbounded face and add edges from u to each vertex of Γ. The resulting graph Γ^* is a triangulation of the plane and consequently has a Grünbaum coloring. In such a coloring the edges incident with u can use either two or three colors. If the edges incident with u use only two colors, then these colors must alternate. Consequently, the colors on the edges of Γ must all be the same. If there are three colors used on the edges incident with u, then one color must be repeated on opposite edges, and the other two colors used on one edge each. In this case the colors on the edges of Γ must occur in consecutive pairs. This can happen in only two different ways up to color permutation. A priori we don't know which of these colorings of the edges incident with u and bounding Γ extend to the interior of Γ^*. At least one such coloring must extend since Γ^* has a Grünbaum coloring. We finish the proof by exhibiting in Figure 2 three different edge colorings of H_7+K_2, one for each possible edge coloring of the edges in Γ. □

Lemma 3.5. *$C_3 + C_5$ is uniquely embeddable in the torus.*

Proof. Proceeding as in Lemma 3.3 we consider a spanning θ-subgraph of C_3+C_5. Denote the vertices of C_3 by $\{a,b,c\}$ and those of C_5 by $\{1,2,3,4,5\}$. Choose the spanning θ-subgraph G_θ to have the vertices $\{a,c\}$ and the three paths abc, $a123c$, $a54c$. We will again have exactly one quadrilateral face and the rest triangles, and again we provide sample cases.

Cases I, II. The two potential planar embeddings with either path of C_5 as the middle path of G_θ each induce a 5-sided face and are therefore impossible.

Case III. The 2-cell embedding of G_θ on the torus extends uniquely to an embedding of $C_3 + C_5$. $\square$

Lemma 3.6. *If G is a triangulation of the torus that contains C_3+C_5, then G has a Grünbaum coloring.*

Proof. Since C_3+C_5 has a unique embedding with exactly one non-triangular face that happens to be 4-sided, the proof follows that of Lemma 3.4. The three different edge colorings of $C_3 + C_5$ that we need to complete the proof are presented in Figure 2. $\square$

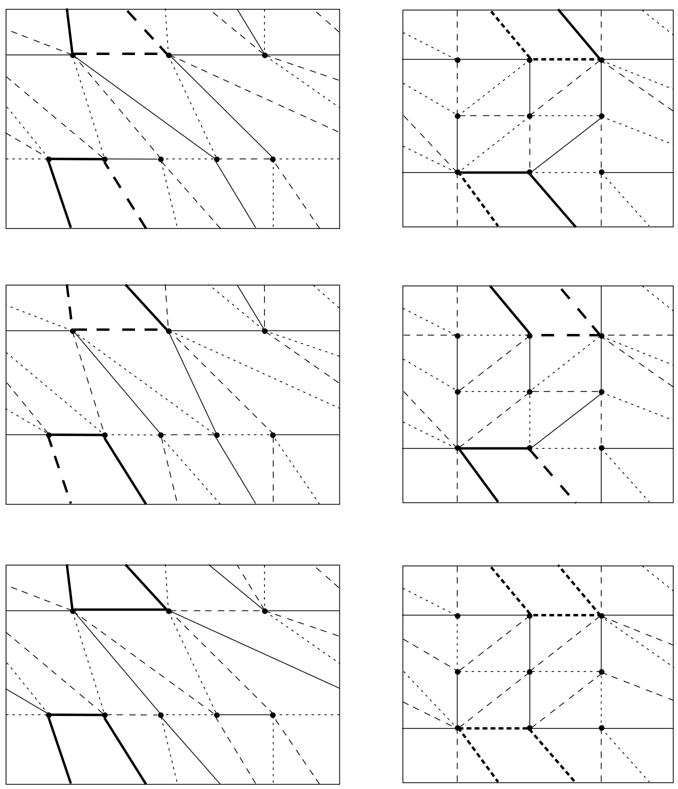

Figure 2: Colorings of $C_3 + C_5$ and $H_7 + K_2$.

4 Six-Chromatic Toroidal Triangulations, II

K_6 has 4 embeddings (three shown in Figure 3, and the last in Figure 8), which are identified by their non-triangular faces: $(4,4,4)_A$ has three squares forming a laddered cylinder, $(4,4,4)_B$ has three squares forming a non-laddered cylinder, $(5,4)$ has one pentagonal and one square face, and (6) has one hexagon.

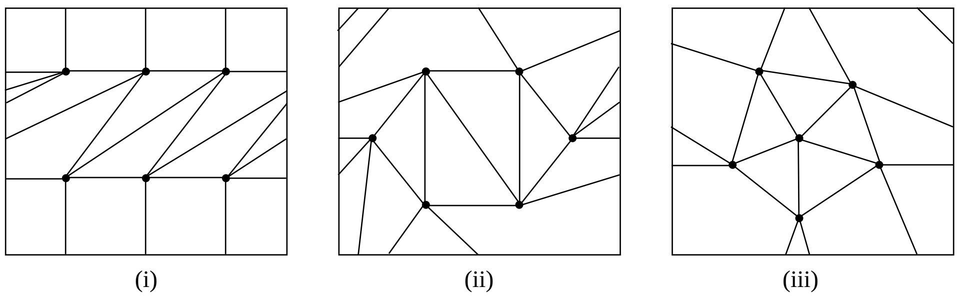

Figure 3: Three toroidal embeddings of K_6: (i) $(4,4,4)_A$, (ii) $(4,4,4)_B$, and (iii) $(5,4)$.

Suppose G is an embedded graph. A 3-coloring of a subset of the edges of G is called a *partial Grünbaum coloring* if the edges of every facial triangle are assigned three distinct colors. Note that if we have a (partial) Grünbaum coloring of an embedded graph G, then we can use Kempe chain arguments to produce alternative (partial) Grünbaum colorings. Formally, we consider G^*, the dual embedding of G on the torus. Note that G^* is a cubic graph. If a particular edge, say e, is colored p, then a $p-t$ *Kempe chain* at e is the set of edges that are colored either p or t and correspond with the standard edge Kempe chain in G^* containing e. A $p-t$ *Kempe change* at e switches the colors in the $p-t$ Kempe chain at e.

Lemma 4.1 ([10])**.** Suppose Γ is a separating square in G, a triangulation of the plane. In any Grünbaum coloring of G each color appears an even number of times on Γ.

Proof. In any Grünbaum coloring of a triangulation of any surface every Kempe chain is a cycle. Thus each edge of Γ is in a Kempe chain with some other edge of Γ. Since there are three colors used on four edges of Γ, at least one color must appear an even number of times. Call this color t. Suppose color p appears an odd number of times on Γ. Consider the $t-p$ Kempe chains. They are disjoint, and each contains an even number of edges of Γ. However, the total number of edges in Γ colored either t or p is odd, a contradiction. $\square$

The preceding argument can be easily modified to give the following.

Lemma 4.2. *Suppose* Γ *is a separating* $2n$*-gon (resp.* $(2n+1)$*-gon) in* G*, a triangulation of the plane. In any Grünbaum coloring of* G *each color appears an even (resp. odd) number of times on* Γ*.*

It follows that any separating square in a Grünbaum-colored triangulation may have only the following types of colorings: *tttt* (denoted C for "constant"), *tptp* (denoted A for "alternating"), *ttpp* (denoted B_1), and *tppt* (denoted B_2). Note that C and A are invariant under rotations and recoloring. While B_1 and B_2 are equivalent under rotation, the orientation of a square within an embedding may require one and exclude the other.

Lemma 4.3 ([10])**.** *In the plane, a triangulation of the interior of a square has a partial Grünbaum coloring of type* B_1 *or* B_2 *if and only if it also has a partial Grünbaum coloring of type A or C.*

Proof. We give one direction of one case of the proof. The other direction follows immediately since Kempe changes are reversible. The second case is identical. Suppose we have a triangulation of the interior of a square Γ. Assume Γ has a partial Grünbaum coloring of type B_1. Consider the two segments of $p-t$ Kempe chains intersecting the edges of Γ. Either one of these intersects Γ in two edges colored t and the other intersects Γ in two edges colored p, or there are two Kempe chains each intersecting Γ in one edge colored t and one edge colored p. In the former case we can make a Kempe change to yield a partial Grünbaum coloring of type C. In the latter case where the edges of Γ are colored *ttpp* we claim that Kempe chains must join consecutive edges; if not, there will be two crossing Kempe chains, contradicting the Jordan Curve Theorem. Then, we can make one Kempe change to create a partial Grünbaum coloring of type A. $\square$

4.1 The $(4,4,4)$ embeddings of K_6.

Lemma 4.4. *Every* 6*-chromatic toroidal triangulation containing the* $(4,4,4)_B$ *embedding of* K_6 *has a Grünbaum coloring.*

Proof. Consider the three squares of the $(4,4,4)_B$ embedding. We enumerate the colorings of this configuration, and show that each possibility is compatible with a partial Grünbaum coloring of $(4,4,4)_B$. Consider three specific exterior triangulations of the square, the 4-wheel and the two ways of adding a diagonal. The first exterior triangulation gives rise to colorings of the square C, B_1, or B_2; one of the added-diagonal exterior triangulations has colorings of the square A or B_1; and the other added-diagonal exterior triangulation has colorings of the square A or B_2. Therefore any triangulated square on a disk may be colored as (A or B_1) and (A or B_2) and (C or B_1 or B_2).

From Lemma 4.3, we know that any square may be colored as (B_1 or B_2) and (A or C). These conditions combine to give the logically equivalent statement that any square on a disk may be colored (A and B_1) or (A and B_2) or (C and B_1 and B_2). Calling the coloring possibilities (A and B_1) type 1, (A and B_2) type 2, and (C and B_1 and B_2) type 3, we have 27 total possibilities for coloring the set of three squares. Note that 111, 112, 121, 211, 221, 212, 122, and 222 may all be colored A A A; 113, 123, 311, 313, and 321 may all be colored B_1 A B_1; 223, 233, 323, and 333 may all be colored B_2 B_2 C; 213, 131, 133, and 231 may be colored A B_1 B_1; 331, 332, 132, and 312 may all be colored B_1 B_1 A; and, 232 and 322 may be colored B_2 B_2 A. It is now straightforward to verify that each of these colorings is compatible with a partial Grünbaum coloring of $(4,4,4)_B$ as shown in Figure 4. □

Lemma 4.5. *Every* 6*-chromatic toroidal triangulation containing the* $(4,4,4)_A$ *embedding of* K_6 *has a Grünbaum coloring.*

Proof. Using the same notation as in the preceding lemma, we have 27 total possibilities for coloring the triple of squares. However, the embedding $(4,4,4)_A$ is rotationally symmetric along the cylinder formed by the triple of squares, so these reduce to 11 possibilities. Note that 113, 133, 233, 231, and 213 may be colored A B_1 B_1; 223 may be colored A B_2 B_2; 111, 112, 122, and 222 may be colored A A A; and, 333 may be colored C C C. It is straightforward to verify that each of these colorings is compatible with a partial Grünbaum coloring of $(4,4,4)_A$ as shown in Figure 5. □

4.2 The $(5,4)$ embedding of K_6.

Lemma 4.6. *Every* 6*-chromatic toroidal triangulation containing the* $(5,4)$ *embedding of* K_6 *has a Grünbaum coloring.*

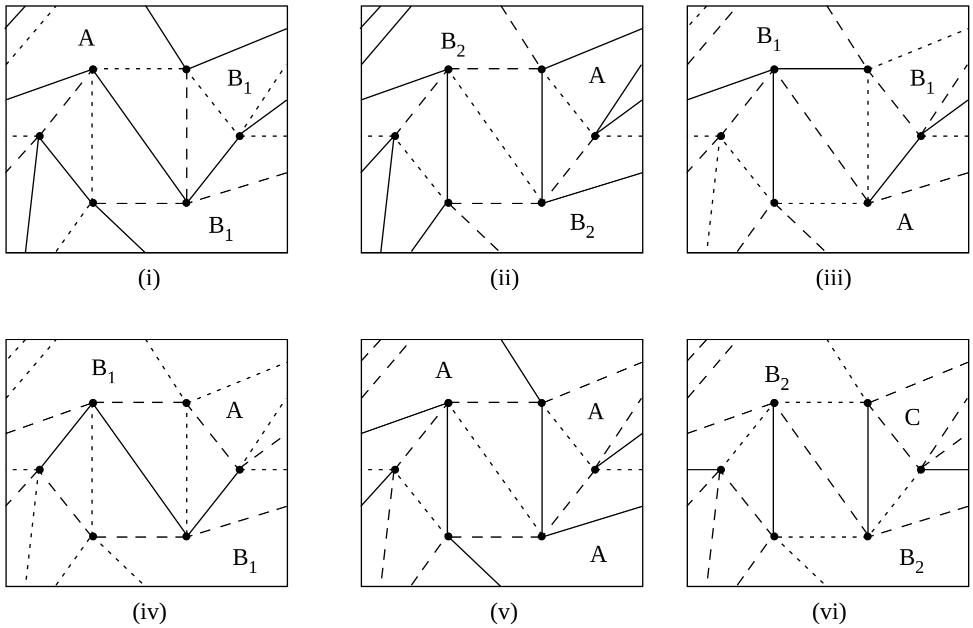

Figure 4: Six partial Grünbaum colorings of $(4,4,4)_B$: (i) A B_1 B_1, (ii) B_2 B_2 A, (iii) B_1 A B_1, (iv) B_1 B_1 A, (v) A A A, and (vi) B_2 B_2 C.

Proof. Consider a triangulation of the torus containing the $(5,4)$ embedding of K_6. We label the edges of the triangulated pentagon as shown in Figure 6. Removing the triangulated pentagon, placing it in the plane, and triangulating its exterior, we obtain a Grünbaum coloring of this triangulation and thus a partial Grünbaum coloring of the triangulated pentagon. We know from Lemma 4.2 that each of the three colors must appear an odd number of times. Thus one color, say t, must appear exactly three times while the other two colors, say p and g, each appear once. We let $j;k$ denote the instance where p is assigned to edge j and g is assigned to edge k. Since the naming of the colors could be reversed, if a coloring is of the form $j;k$, it could also be labeled $k;j$.

Suppose we have a triangulated pentagon in the disk colored so that the three edges colored t are consecutive, i.e., $|k-j|=1$. As in the proof of Lemma 4.3, the $j;j+1$ coloring can be Kempe changed to either $j;j+2$ or $j;j+4$ but not to $j;j+3$. Reversing colors, we see that a triangulated pentagon colored $j;j+4$ may be Kempe changed to one colored $j;j+1$ or

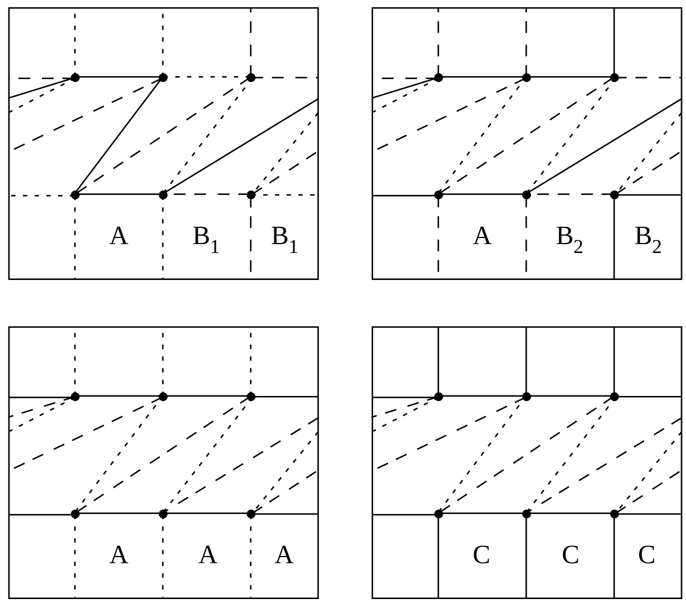

Figure 5: Four partial Grünbaum colorings of $(4,4,4)_A$.

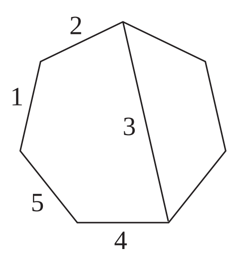

Figure 6: The non-triangular faces of the $(5,4)$ embedding of K_6.

$j; j+3$. Similarly, a triangulated pentagon colored $j; j+2$ may be Kempe changed to one colored $j; j+1$ or $j; j+3$ and a triangulated pentagon colored $j; j+3$ may be Kempe changed to one colored $j; j+2$ or $j; j+4$.

Now suppose we have a triangulation of the torus containing the $(5,4)$ embedding of K_6. It contains a triangulated pentagon whose colorings we have discussed above, and a triangulated square that has coloring type 1 (A and B_1), type 2 (A and B_2), or type 3 (C and B_1 and B_2). Examining the partial Grünbaum colorings of $(5,4)$ in Figure 7, we see that these can be colored with corresponding pentagonal and square colorings, as *tptpptg* $\to$ $2;5$ B_1, *tpgpptt* $\to$ $2;3$ B_1, *tttppgp* $\to$ $4;5$ B_1, *tpptpgt* $\to$ $2;4$ A, *tppgpgg* $\to$ $1;2$ A, and *ttptppg* $\to$ $4;5$ A.

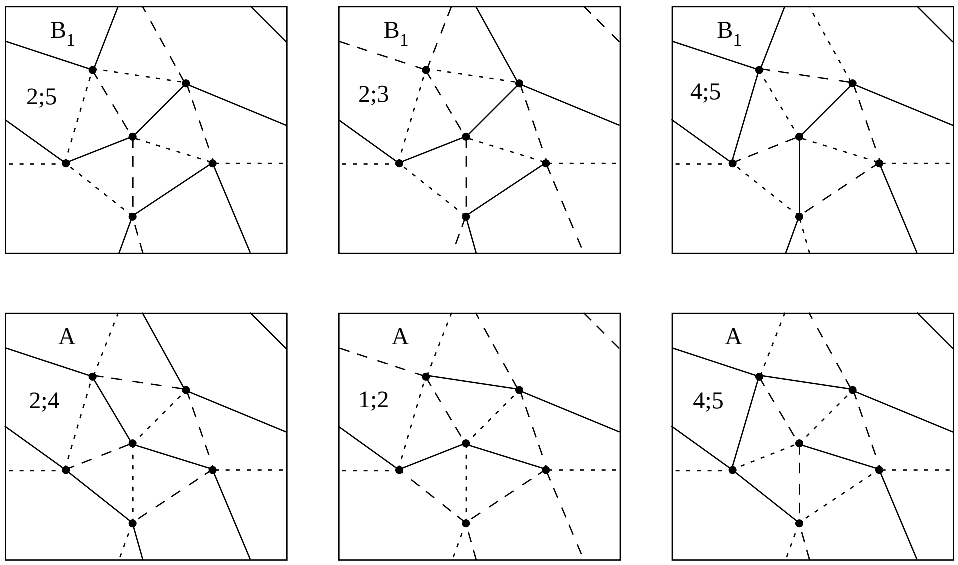

Figure 7: The six required partial Grünbaum colorings of $(5,4)$.

We now argue that any possible triangulation of a pentagon identified along an edge with a square has a partial Grünbaum coloring that can be Kempe changed to one of the six partial Grünbaum colorings listed above. First, suppose that the square has coloring type 1 or 2. It may be colored A, and so a pentagon with a triangulation-induced coloring $2;4$, $1;2$, or $4;5$ will (together with an A-colored square) form a heptagon compatible with a partial Grünbaum coloring of $(5,4)$. We proceed with the remaining 7 $j;k$ colorings of the pentagon, assuming the square is type 1 or 2, by sequentially reducing them to cases already accounted for. A pentagon with coloring $4;1$ can be Kempe changed to be colored $4;5$ or $4;2$, both of which are already known to (together with a type 1 or type 2 square) form a heptagon compatible with a partial Grünbaum coloring of $(5,4)$. Similarly a pentagon with coloring $4;3$ can also be Kempe changed to $4;5$ or $4;2$. Pentagons with coloring $1;3$ or with coloring $1;5$ can be Kempe changed to be colored $1;2$ or $1;4$, both of which are already known to form compatible heptagons. Those with coloring $2;3$ or $2;5$ can be Kempe changed to be colored $2;4$ or $2;1$, both of which are already done. Finally, a pentagon with coloring $3;5$ can be Kempe changed to be colored $3;4$ or $3;1$, both of which are already done. This accounts for all ten possible partial Grünbaum colorings of a pentagon combined with a type 1 or type 2 square to form a heptagon.

Now suppose that the square has coloring type 3. It may be colored B_1, and so a pentagon with triangulation-induced coloring $2;5$, $2;3$, or $4;5$ will (together with a B_1-colored square) form a heptagon compatible with a partial Grünbaum coloring of $(5,4)$. We follow the same type of strategy as above. Pentagons with coloring $2;4$ or $2,1$ can each be Kempe changed to one of $2;3$ or $2;5$. Those with $5;1$ or $5,3$ can be changed to one of $5,2$ and $5;4$. Next, those with $4;1$ or $4;3$ each change to one of $4;5$ and $4;2$. Finally, a pentagon with coloring $1;3$ can be changed to either $1;2$ or $1;4$. This accounts for all ten possible partial Grünbaum colorings of a pentagon combined with a type 3 square to form a heptagon, and completes the proof. □

4.3 The (6) embedding of K_6.

Lemma 4.7. *Every 6-chromatic toroidal triangulation containing the* (6) *embedding of* K_6 *has a Grünbaum coloring.*

Proof. Consider a triangulation of the torus containing the (6) embedding of K_6. The hexagonal face of (6) is a triangulated hexagon; this may be removed, placed in the plane, and by triangulating the exterior we see that the resulting graph has a Grünbaum coloring and thus the triangulated hexagon has a partial Grünbaum coloring. By Lemma 4.2, the number of hexagon edges in each color must be even. These may be allocated to g, p, and t as 2, 2, 2, or 0, 2, 4, or 0, 0, 6.

Because K_7 has a unique embedding on the torus, and (6) is merely K_7 with one vertex removed, (6) is dihedrally symmetric outside of the hexagonal face. Thus, we can consider partial Grünbaum colorings up to rotation and reflection; the orientation of a triangulation inside the hexagonal face can determine the colors on the edges of the hexagon, which, if they match any rotation/reflection of a partial Grünbaum coloring of (6), will determine a coloring of the remainder of the triangulation.

Let us list all possible partial Grünbaum colorings of a hexagon, up to rotations and reflections. We have *pppppp*, *ttpppp*, *tptppp*, *tpptpp*, *ttppgg*, *ttpgpg*, *tpgtgp*, *tpgtpg*, and *tpptgg*. Of these, *ttpppp*, *tpptpp*, *ttppgg*, and *tpgtpg* appear as part of partial Grünbaum colorings of (6), shown in Figure 8. It remains to be shown that if a partial Grünbaum coloring of a triangulated hexagon induces any of the other five colorings, there exist Kempe changes that render the coloring equivalent to one of the four hexagonal subsets of partial Grünbaum colorings of (6).

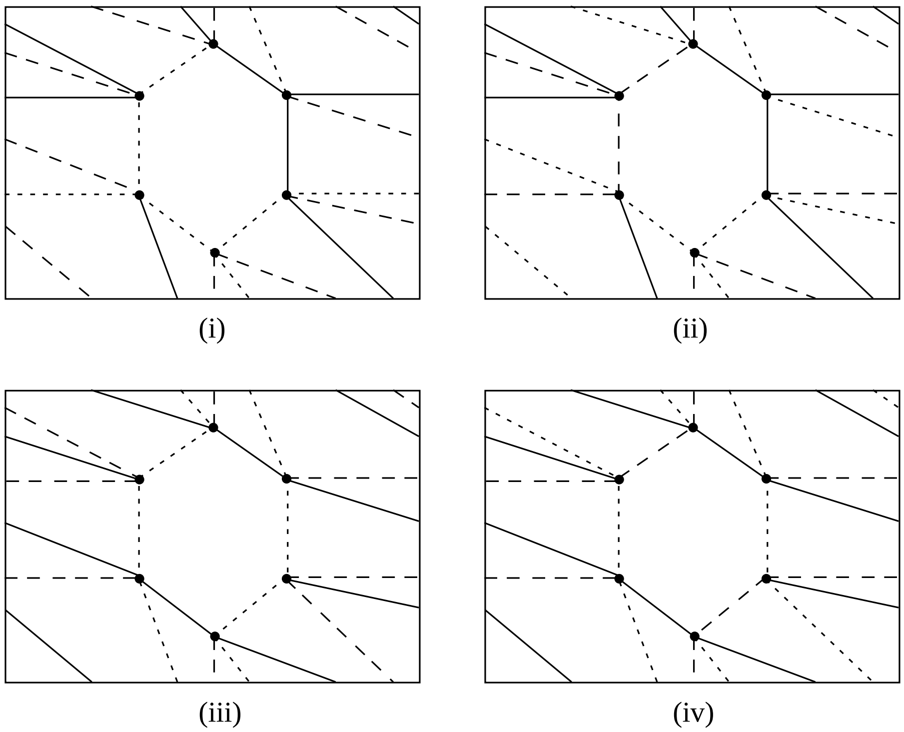

Figure 8: Four partial Grünbaum colorings of (6), (i) *ttpppp*, (ii) *ttppgg*, (iii) *tpptpp*, and (iv) *tpgtpg*.

Now, note that if two colors use exactly 4 hexagon edges, occurring twice each or with all four edges of one color and none of the other, then these four edges behave as those of a square. That is, the four edges are colored A, C, B_1, or B_2. Thus, the proof of Lemma 4.3 holds, and we may conclude that Kempe chains connect only adjacent edges (where adjacency is considered relative to the "square") and not opposing edges (where again, adjacency is considered relative to the "square").

Consider each of the five remaining colorings in turn. For each make a Kempe change on the first p edge. The same argument as Lemma 4.3 determines the possible new colorings. These will either be, or color permute to, a coloring already determined compatible with a partial Grünbaum coloring of (6).

If a partial Grünbaum coloring of a triangulated hexagon induces the coloring *tpgtgp* on the hexagon, make a $p-g$ Kempe change on the first p edge to obtain *tgptgp*, which color-permutes to *tpgtpg*, or to obtain *tggtgg* which color-permutes to *tpptpp*. For the induced hexagon coloring *tpptgg*, make a $p-g$ Kempe change on the first p edge to obtain *tggtgg*, or to obtain *tgptgp*, which color-permutes to *tpgtpg*. For the induced hexagon coloring *ttpgpg*, make a $p-g$ Kempe change on the first p edge to obtain *ttgppg*, which color-permutes to *ggtppt*, which in turn reflects to *tpptgg*, or obtain *ttggpp*, which color-permutes to *ttppgg*. For the induced hexagon coloring *tptppp*, make a $p-g$ Kempe change on the first p edge to obtain *tgtgpp*, which color-permutes to *gpgptt*, which in turn reflects to *ttpgpg*, or obtain *tgtppg*, which color-permutes to *gpgttp* and rotates to *ttpgpg*. Finally, for the induced hexagon coloring *pppppp*, make a $p-t$ Kempe change on the first p edge to obtain one of *ttpppp*, *tptppp*, or *tpptpp*. The colorings *ttpppp* and *tpptpp* are compatible with partial Grünbaum colorings of (6), and *tptppp* was dealt with above. □

Acknowledgments. Thanks to sarah-marie belcastro's topological graph theory class at the 2007 session of the Hampshire College Summer Studies in Mathematics (particularly Miles Edwards and Nate Harman), for enthusiastically working on open problems, some of whose solutions appear here.

References

[1] M. O. Albertson, J. P. Hutchinson, The independence ratio and genus of a graph, Trans. Amer. Math. Soc. 226 (1977) 161–173.

[2] M. O. Albertson, J. P. Hutchinson, The maximum size of an independent set in a nonplanar graph, Bull. AMS 81 (1975) 554–555.

[3] M. O. Albertson, J. P. Hutchinson, On six-chromatic toroidal graphs, Proc. London. Math. Soc. Third Series XLI (1980) 533–556.

[4] A. Altshuler, Construction and enumeration of regular maps on the torus, Discrete Math. 4 (1973), 201–217.

[5] K. Appel, W. Haken, J. Koch, Every Planar Map is Four Colorable. I: Discharging,Illinois J. Math. 21 (1977) 429–490.

[6] K. Appel, W. Haken, Every Planar Map is Four-Colorable, II: Reducibility, Illinois J. Math. 21, (1977) 491-567.

[7] D. Archdeacon, Problems in Topological Graph Theory: Three-Edge-Coloring Orientable Triangulations,
`http://www.emba.uvm.edu/~archdeac/problems/grunbaum.htm`.

[8] N. L. Biggs, E. . Lloyd, R. J. Wilson, Graph Theory 1736–1936, Clarendon Press, Oxford, 1976.

[9] G. Dirac, Map-colour theorems, Canadian J. Math. 4, (1952). 480–490.

[10] M. Edwards, N. Harman, personal communication, July 2007.

[11] A. Gagarin, W. Kocay, D. Neilson, Embeddings of small graphs on the torus, Cubo Mat. Educ. 5 (2003), no. 2, pp. 351–371.

[12] B. Grünbaum, Conjecture 6, in: W.T. Tutte (Ed.), Recent Progress in Combinatorics, Academic Press, New York, 1969, p.343.

[13] T. Jensen, B. Toft, Graph Coloring Probelms, Wiley Interscience Series, New York, 1975.

[14] M. Kochol, personal communication, October 2007.

[15] B. Mohar, C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, 2001.

[16] N. Robertson, D. Sanders, P. Seymour, R. Thomas, The four-colour theorem, J. Combinatorial Theory Ser. B 70 (1997), no. 1, 2–44.

[17] C. Thomassen, Five-coloring graphs on the torus, J. Combin. Theory Ser. B 62 (1994), 11–33.