



Contents

Preface for Instructors and Other Teachers	xvii
1 About This Book	xvii
2 How to Use This Book	xx
2.1 A Start on Discovery-Based Learning	xxi
2.2 Details of Conducting Group Work	xxiii
3 Chapter and Bonus-Section Dependencies	xxvi
Preface for Students and Other Learners	xxix
1 About This Book (and about Learning Mathematics)	xxix
2 How to Use This Book	xxx
2.1 How to Use This Book in a Class	xxxii
2.2 How to Use This Book for Self-Study	xxxii
3 Tips for Reading Mathematics	xxxiii
4 Problem-Solving Prompts	xxxiv
5 Tips for Writing Mathematics	xxxv
Acknowledgments	xxxix
I Theme: The Basics	1
1 Counting and Proofs	3
1.1 Introduction and Summary	3
1.2 Try This! Let's Count	3
1.3 The Sum and Product Principles	6
1.4 Preliminaries on Proofs and Disproofs	9
1.5 Pigeons and Correspondences	14
1.6 Where to Go from Here	19
1.7 Problems That Use Counting or Proofs	20
1.8 Instructor Notes	22

2	Sets and Logic	25
2.1	Introduction and Summary	25
2.2	Sets	26
2.2.1	Making New Sets from Scratch	27
2.2.2	Finding Sets inside Other Sets	28
2.2.3	Proof Technique: Double-Inclusion	29
2.2.4	Making New Sets from Old	30
2.2.5	Looking at Sets	31
2.3	Logic	35
2.3.1	Combining Statements	36
2.3.2	Restriction of Variables via Quantifiers	40
2.3.3	Negation Interactions	42
2.4	Try This! Problems on Sets and Logic	44
2.5	Proof Techniques: Not!	46
2.6	Try This! A Tricky Conundrum	48
2.7	Where to Go from Here	49
2.8	Bonus: Truth Tellers	50
2.9	Problems about Sets and Logic	53
2.10	Instructor Notes	55
3	Graphs and Functions	57
3.1	Introduction and Summary	57
3.2	Function Introduction	57
3.3	Try This! Play with Functions and Graphs	63
3.3.1	Play with Functions	63
3.3.2	Play with Graphs	63
3.3.3	A Dot Game	65
3.4	Functions and Counting	66
3.5	Graphs: Definitions and Examples	67
3.6	Isomorphisms	71
3.7	Graphs: Operations and Uses	74
3.7.1	Sets and Graphs Have Some Things in Common	74
3.7.2	How Are Graphs Useful?	76
3.8	Try This! More Graph Problems	78
3.9	Ramseyness	80
3.10	Where to Go from Here	81

3.11 Bonus: Party Tricks	83
3.12 Bonus 2: Counting with the Characteristic Function	84
3.13 Problems about Graphs and Functions	85
3.14 Instructor Notes	88
4 Induction	91
4.1 Introduction and Summary	91
4.2 Induction	91
4.2.1 Summation Notation	98
4.2.2 Induction Types and Styles	99
4.3 Try This! Induction	100
4.4 More Examples	101
4.5 The Best Induction Proof Ever	103
4.6 Try This! More Problems about Induction	104
4.7 Are They or Aren't They? Resolving Grey Ducks	105
4.8 Where to Go from Here	106
4.9 Bonus: Small Crooks	107
4.10 Bonus 2: An Induction Song	108
4.11 Problems That Use Induction	109
4.12 Instructor Notes	111
4.12.1 Potential Practice Proof Problems	112
5 Algorithms with Ciphers	115
5.1 Introduction and Summary	115
5.2 Algorithms	115
5.2.1 Conditionals and Loops	119
5.2.2 Efficiency	123
5.2.3 Algorithms and Existence Proofs	124
5.3 Modular Arithmetic (and Equivalence Relations)	126
5.4 Cryptography: Some Ciphers	131
5.4.1 Shift Ciphers	131
5.4.2 Atbash Ciphers	133
5.4.3 The Vigenère Cipher	133
5.4.4 Decryption and the Real World	136
5.5 Try This! Cryptoequivalent Modular Algorithmic Problems	137
5.6 Where to Go from Here	138

5.7	Bonus: Algorithms for Searching Graphs	140
5.8	Bonus 2: Pigeons and Divisibility	142
5.9	Problems about Algorithms, Modular Arithmetic, and Ciphers	145
5.10	Instructor Notes	148
II	Theme: Combinatorics	151
6	Binomial Coefficients and Pascal's Triangle	153
6.1	Introduction and Summary	153
6.2	You Have a Choice	153
6.3	Try This! Investigate a Triangle	156
6.4	Pascal's Triangle	158
6.5	Overcounting Carefully and Reordering at Will	160
6.6	Try This! Play with Powers and Permutations	164
6.7	Binomial Basics	165
6.8	Combinatorial Proof	168
6.9	Try This! Pancakes and Proofs	169
6.10	Where to Go from Here	171
6.11	Bonus: Sorting Bubbles in Order of Size	172
6.12	Bonus 2: Mastermind	175
6.12.1	One Strategy for Playing	176
6.12.2	Mini-Project	178
6.13	Problems Binomially Combinatorial in Nature	180
6.14	Instructor Notes	183
7	Balls and Boxes and PIE: Counting Techniques	185
7.1	Introduction and Summary	185
7.2	Combinatorial Problem Types	185
7.3	Try This! Let's Have Some PIE	192
7.4	Combinatorial Problem Solutions and Strategies	193
7.4.1	Strategy: Slots	193
7.4.2	Strategy: Stars and Bars	194
7.4.3	Solutions to Problem Types	196
7.4.4	Denouement: Bijective Counting, Again	201
7.5	Let's Explain Our PIE!	204

7.6	Try This! What Are the Balls and What Are the Boxes? And Do You Want Some PIE?	207
7.7	Where to Go from Here	209
7.8	Bonus: Linear and Integer Programming	209
7.9	Problems about Balls, Boxes, and PIEs	214
7.10	Instructor Notes	219
8	Recurrences	221
8.1	Introduction and Summary	221
8.2	Fibonacci Numbers and Identities	221
8.3	Recurrences and Integer Sequences and Induction	224
8.4	Try This! Sequences and Fibonacci Identities	229
8.5	Naive Techniques for Finding Closed Forms and Recurrences	230
8.6	Arithmetic Sequences and Finite Differences	231
8.7	Try This! Recurrence Exercises	234
8.8	Geometric Sequences and the Characteristic Equation	235
8.9	Try This! Find Closed Forms for <i>These</i> Recurrence Relations!	241
8.10	Where to Go from Here	241
8.11	Bonus: Recurring Stories	242
8.12	Recurring Problems	246
8.13	Instructor Notes	249
9	Cutting Up Food: Counting and Geometry	251
9.1	Introduction and Summary	251
9.2	Try This! Slice Pizza (and a Yam)	251
9.3	Pizza Numbers	253
9.4	Try This! Spaghetti, Yams, and More	256
9.5	Yam, Spaghetti, and Pizza Numbers	258
9.5.1	Let's Go for It! Hyperbeet Numbers	260
9.6	Where to Go from Here	263
9.7	Bonus: Geometric Gems	264
9.8	Problems That Combine Combinatorial Topics	267
9.9	Instructor Notes	272

III	Theme: Graph Theory	275
10	Trees	277
	10.1 Introduction and Summary	277
	10.2 Basic Facts about Trees	277
	10.3 Try This! Spanning Trees	280
	10.4 Spanning Tree Algorithms	282
	10.4.1 Greedy Algorithms	290
	10.5 Binary Trees	292
	10.6 Try This! Binary Trees and Matchings	297
	10.7 Matchings	299
	10.8 Backtracking	300
	10.9 Where to Go from Here	303
	10.10 Bonus: The Branch-and-Bound Technique in Integer Programming	304
	10.11 Tree Problems	306
	10.12 Instructor Notes	310
11	Euler's Formula and Applications	313
	11.1 Introduction and Summary	313
	11.2 Try This! Planarity Explorations	313
	11.3 Planarity	315
	11.4 A Lovely Story	316
	11.5 Or, Are Emus Full?: A Theorem and a Proof	318
	11.6 Applications of Euler's Formula	320
	11.7 Try This! Applications of Euler's Formula	323
	11.8 Where to Go from Here	325
	11.9 Bonus: Topological Graph Theory	325
	11.10 Problems about Planar Graphs	328
	11.11 Instructor Notes	331
12	Graph Traversals	333
	12.1 Introduction and Summary	333
	12.2 Try This! Euler Traversals	333
	12.3 Euler Paths and Circuits	335

- 12.4 Dijkstra’s Algorithm, with sides of Hamilton Circuits and the
Traveling Salesperson Problem 339
- 12.5 Try This!—Do This!—Try This! 343
- 12.6 Where to Go from Here 345
- 12.7 Bonus: Digraphs, Euler Traversals, and RNA Chains 346
- 12.8 Bonus 2: Network Flows 348
- 12.9 Bonus 3: Two Hamiltonian Theorems 353
- 12.10 Problems with Traversing 354
- 12.11 Instructor Notes 360
- 13 Graph Coloring 361
 - 13.1 Introduction and Summary 361
 - 13.2 Try This! Coloring Vertices and Edges 361
 - 13.2.1 Vertex Coloring 361
 - 13.2.2 Edge Coloring 362
 - 13.2.3 More on Vertex Coloring 363
 - 13.2.4 More on Edge Coloring 363
 - 13.3 Introduction to Coloring 364
 - 13.3.1 Coloring Bounds 367
 - 13.3.2 Applications of Vertex Coloring 369
 - 13.4 Try This! Let’s Think about Coloring 373
 - 13.5 Coloring and Things (Graphs and Concepts) That Have Come
Before 375
 - 13.5.1 Let’s Color the Edges of Complete Graphs 375
 - 13.5.2 Let’s Color Bipartite Graphs 376
 - 13.5.3 Add a Condition, Get a Different Bound 378
 - 13.5.4 Greedy Matchings 379
 - 13.6 Where to Go from Here 380
 - 13.7 Bonus: The Four-Color Theorem 381
 - 13.8 Colorful Problems 386
 - 13.9 Instructor Notes 390
- IV Other Material 393
- 14 Probability and Expectation 395
 - 14.1 Introduction and Summary 395
 - 14.2 What *Is* Probability, Exactly? 396

14.3	High Expectations	398
14.4	You Are Probably Expected to Try This!	405
14.5	Conditional Probability and Independence	406
14.5.1	The Helpfulness of PIE in the Real World of Probability	409
14.5.2	Independence versus Exclusivity	411
14.6	Try This! . . . , Probably, Under Certain Conditions	413
14.7	Higher Expectations	415
14.7.1	That's <i>Wild!</i> (A Hint at the Probabilistic Method)	416
14.8	Where to Go from Here	418
14.9	Bonus: Ramsey Numbers and the Probabilistic Method	419
14.10	Expect Problems, Probably	422
14.11	Instructor Notes	427
15	Fun with Cardinality	429
15.1	Introduction and Summary	429
15.2	Read This! Parasitology, the Play	429
15.2.1	Scene 1: The Storage Coordinator	430
15.2.2	Scene 2: The Taxonomist	435
15.2.3	Scene 3: The Café	437
15.2.4	Scene 4: Cataloguing	442
15.3	How Big Is Infinite?	445
15.4	Try This: Investigating the Play	446
15.4.1	Questions about Sample Storage	446
15.4.2	More Questions about Sample Storage	447
15.4.3	Questions about Café Conversations	448
15.4.4	Indiscrete Questions	449
15.5	How High Can We Count?	450
15.5.1	The Continuum Hypothesis	453
15.6	Where to Go from Here	453
15.7	Bonus: The Schröder–Bernstein Theorem	454
15.8	Infinitely Large Problems	455
15.9	Instructor Notes	457
A	Additional Problems	459
B	Solutions to Check Yourself Problems	487

Contents	xv
C The Greek Alphabet and Some Uses for Some Letters	517
D List of Symbols	519
Glossary	523
Bibliography	537